Mortar spectral element discretization of Darcy's equations in nonhomogeneous medium

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Abstract : We consider Darcy's equations with piecewise continuous coefficients in a bounded two-dimensional domain. We propose a spectral element discretization of this problem wich relies on the mortar domain decomposition technique. We prove optimal error estimates. We also performe numerical analysis of the discrete problem and present numerical experiments. They turn out to be in good coherency with the theoretical results.

Résumé : Les équations de Darcy modélisent l'écoulement d'un fluide visqueux incompressible dans un milieu poreux rigide. Un des paramètres dépend de la perméabilité du milieu et, lorsqu'il n'est pas homogène, les variations de ce paramètre peuvent être extrêmement importantes. Pour traiter ce phénomène, nous proposons une discrétisation du modèle par éléments spectraux avec joints, l'idée étant de construire une décomposition du domaine telle que la perméabilité soit constante sur chaque élément de la partition. Nous effectuons l'analyse a priori de cette discrétisation et présentons quelques expériences numériques qui confirment les résultats de l'analyse.

Keywords : Mortar spectral elements, discontinuous coefficients, Darcy's equations.

1 Introduction

This paper is devoted to the analysis of the mortar spectral element discretization of the problem introduced by Darcy [14],

$$\begin{cases} \alpha \boldsymbol{u} + \mathbf{grad}\, p = \alpha \boldsymbol{f}, & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

in a bounded two-dimensional domain Ω with a Lipschitz-continuous boundary $\partial\Omega$, and let $\boldsymbol{n}$ denote the unit outward normal vector to Ω on $\partial\Omega$. The function α is given with positive values. We are interested in the case where this function is not globally continuous but only piecewise smooth and also such that the ratio of the maximal value to its minimal value is large. This models, for instance, the flow of a viscous incompressible fluid in a rigid porous inhomogeneous medium.

In a first step, we consider the key situation where the function α is piecewise constant. The discretization of this problem by mortar finite element discretization is studied in [9], and optimal a priori and a posteriori error estimates are proven. But the idea of this paper is different : the discretization rely on a domain decomposition such that, on each subdomain, the function α is constant. To this end, the mortar element technique, introduce in [11], seems especially appropriate since it allows for working with noncorforming decompositions, i.e. the intersection of two subdomains is not restricted to be a corner or a whole edge of both of them. A consequence of this property, in the present situation, is that the number of subdomains in order to take into account the discontinuities of α can be highly reduced. We refer to [17, Subsec. 1.5] for a first application of this method to discontinuous coefficients in the finite element framework.

Here, on each subdomain, we consider a spectral discretization. As is well known, spectral and spectral element methods rely on the approximation by high degree polynomials and on the use of tensorized bases of polynomials. For these reasons, the basic geometries are rectangles. Even if these methods can easily be extended to convex or curved quadrilaterals, the arguments for such an extension are very technical, so we have rather avoid them in this paper. For this reasons, the subdomains that we consider are only rectangles, and we refer the reader to [16] for the treatment of more complex geometries for a simpler problem. It was extended [3] to the bilaplacian equation where the variational space is the standard space $H^2(\Omega)$ of functions with square-integrable first-order and second-order derivatives and also to the Stokes problem which is of saddle-point type, however it still involves usual Sobolev spaces. We also quote [2] for an application of the mortar technique to weighted Sobolev spaces, in order to handle discontinuous boundary conditions for the NavierStokes equations.

Another advantage of the mortar method is that it allows for working with independent discretization parameters on the subdomains. Our idea here is to use different degrees of polynomials on these subdomains, in order to take into account the different values of α. Indeed, in practical situations, even the ratio of the values of α on adjacent subdomains can be high, and the intuitive idea is to take higher degrees of polynomials in the subdomains where α is large. We perform the numerical analysis of this discretization, in order to optimize the choice of the degrees of polynomials on each subdomains as a function of the value of α and also of the

geometry of the domain, since the geometrical singular functions issued from the non-convex corners of the domains interfere with the singularities issued from the discontinuities of α.
We also present the extension of the discretization to the case of a piecewise smooth function α : this comes either from the thermic properties of the medium where the permeability coefficient can depend on the density or from transformation of the geometry, for instance if the coefficients are piecewise constant on convex quadrilaterals. Since handling smooth coefficient is standard in spectral methods, the only difficulty here is to preserve the efficiency of the algorithm for solving the corresponding discrete system.
Finally, the implementation of the mortar technique mainly relies on an appropriate treatment of the matching conditions on the interfaces that we briefly describe (we refer to [4] for another way of handling these conditions). We describe some numerical experiments, which are in good coherency with the analysis and justify the choice of a domain decomposition technique and the use of different degrees of polynomials.
The outline of this paper is as follows. In Section 2, we briefly recall some properties of the continuous problem. Section 3 is devoted to the numerical analysis of the mortar spectral element discretization of the problem in the case where the function α is piecewise constant. Error estimates between the exact and discrete solutions are established in Section 4.These results are extended to the case of piecewise smooth functions in Section 4. In Section 5, we present some numerical experiments in order to compare the present method with the spectral discretization without domain decomposition or a conforming spectral element discretization.

2 The continuous problem

Let Ω be a bounded connected open set in $\mathbb{R}^2$, with a Lipschitz-continuous boundary $\partial\Omega$. Throughout the paper, we make the following assumptions on the function α : there exists a finite number of domains Ω_k^*, $1 \leq k \leq K^*$, such that

- they form a partition of Ω without overlapping

$$\overline{\Omega} = \cup_{k=1}^{K^*} \overline{\Omega}_k^* \text{ and } \Omega_k^* \cap \Omega_{k'}^* = \emptyset,\ 1 \leq k < k' \leq K^*, \tag{2}$$

- the restriction of α to each $\overline{\Omega}_k^*$, $1 \leq k \leq K^*$, is continuous on Ω_k^*,
- the restriction of α to each $\overline{\Omega}_k^*$, $1 \leq k \leq K^*$, is bounded and positive, i.e. there exist constants $\alpha_k^{\max}$ and $\alpha_k^{\min}$ such that

$$\alpha_k^{\max} = \sup_{\boldsymbol{x} \in \overline{\Omega}_k^*} \alpha(\boldsymbol{x}) < +\infty, \quad \text{and } \alpha_k^{\min} = \inf_{\boldsymbol{x} \in \overline{\Omega}_k^*} \alpha(\boldsymbol{x}) > 0. \tag{3}$$

We set

$$\alpha_{\max} = \max_{1 \leq k \leq K^*} \alpha_k^{\max} \text{ and } \alpha_{\min} = \min_{1 \leq k \leq K^*} \alpha_k^{\min}. \tag{4}$$

We define $H^{\frac{1}{2}}(\partial\Omega)$ as the space of traces of functions of $H^1(\Omega)$ on $\partial\Omega$, provided with the trace norm, and $H^{-\frac{1}{2}}(\partial\Omega)$ as its dual space. As usual, $L_0^2(\Omega)$ stands for the space of functions in $L^2(\Omega)$ with a null integral on Ω. Finally, we consider the space $\mathcal{C}^\infty(\overline{\Omega})$ of infinitely differentiable functions on $\overline{\Omega}$ and its subspace $\mathcal{D}(\Omega)$ of functions with a compact support in Ω.
As now well-known (see [13, XIII.1]), system (1) admits several variational formulations. We have chosen the formulation which seems the more convenient in view of the spectral element discretization. So, we consider the variational problem

Find $(\boldsymbol{u}, p)$ in $L^2(\Omega)^2 \times (H^1(\Omega) \cap L_0^2(\Omega))$ such that

$$\begin{cases} \forall \boldsymbol{v} \in L^2(\Omega)^2, a_\alpha(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) & = \displaystyle\sum_{k=1}^{K^*} \int_{\Omega_k^*} \alpha(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \\ \forall q \in H^1(\Omega) \cap L_0^2(\Omega), b(\boldsymbol{u}, q) & = \langle g, q \rangle_{\partial\Omega}, \end{cases} \tag{5}$$

where $\langle ., . \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, while the bilinear forms $a_\alpha(.,.)$ and $b(.,.)$ are defined by

$$a_\alpha(\boldsymbol{u}, \boldsymbol{v}) = \sum_{k=1}^{K^*} \int_{\Omega_k^*} \alpha(\boldsymbol{x}) \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}, \quad b(\boldsymbol{v}, q) = \int_\Omega (\mathbf{grad}\, q)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{6}$$

In order to optimize the constants in all that follows, we introduce the α-dependent norms

$$\|\boldsymbol{v}\|_\alpha = \left(\sum_{k=1}^{K^*} \int_{\Omega_k^*} \alpha(\boldsymbol{x}) |\boldsymbol{v}(\boldsymbol{x})|^2\, d\boldsymbol{x} \right)^{\frac{1}{2}}, \quad |q|_{\alpha*} = \left(\sum_{k=1}^{K^*} \int_{\Omega_k^*} \frac{1}{\alpha(\boldsymbol{x})} |\mathbf{grad}\, q(\boldsymbol{x})|^2\, d\boldsymbol{x} \right)^{\frac{1}{2}}. \tag{7}$$

The fact that the semi-norm $||_{\alpha_*}$ is a norm on $H^1(\Omega) \cap L^2_0(\Omega)$ results from a generalized Bramble-Hilbert inequality and can easily be derived thanks to the Peetre-Tartar lemma, see [15, Chap. I, Thm 2.1].
The well-posedness of this problem was established in [9].

Proposition 1 *For any data* $(\mathbf{f}, g)$ *in* $L^2(\Omega)^2 \times H^{-\frac{1}{2}}(\partial\Omega)$*, problem* (5) *has a unique solution* $(\boldsymbol{u}, p)$ *in* $L^2(\Omega)^2 \times (H^1(\Omega) \cap L^2_0(\Omega))$*. Moreover, this solution satisfies*

$$\|\boldsymbol{u}\|_\alpha + |p|_{\alpha^*} \leq 3(\sqrt{\alpha_{\max}}\|\boldsymbol{f}\|_{L^2(\Omega)^2} + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)}). \tag{8}$$

We are also interested with the regularity properties of this solution. We recall a result witch is proven in Prop. 2.5 of [9].

Proposition 2 *There exists a real number* s_α*,* $0 < s_\alpha < \frac{1}{2}$*, such that the mapping :* $(\mathbf{f}, g) \longmapsto (\boldsymbol{u}, p)$*, where* $(\boldsymbol{u}, p)$ *is the solution of problem* (1)*, is continuous from* $H^s(\Omega)^2 \times H^{s-\frac{1}{2}}(\partial\Omega)$ *into* $H^s(\Omega)^2 \times H^{s+1}(\Omega)$*, pour tout* $s \leq s_\alpha$*.*

Remark 3 *We can exhibit a maximal value* s_α *only limited by*

$$s_\alpha < \min\{\frac{1}{2}, c_\Omega |\log(1 - \frac{\alpha_{\min}}{\alpha_{\max}})|\}, \tag{9}$$

where the constant c_Ω *depends only on the geometry of* Ω*.*

3 Analysis of the Mortar Spectral Element Discretization

Throughout this section, we work with a piecewise constant function α. We introduce a new partition of the domain without overlapping

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \ \text{ and } \Omega_k \cap \Omega_{k'} = \emptyset,\ 1 \leq k < k' \leq K, \tag{10}$$

such that the function α is constant on each Ω_k, $1 \leq k \leq K$ (so, each Ω_k is contained in an Ω_k^*), and also that the Ω_k, $1 \leq k \leq K$, are rectangles. Note that K can be much larger than K^* in order to take into account the geometry of the discontinuities of α.
The decomposition is conforming said to be means that the intersection of two different $\overline{\Omega}_k$, if not empty, is a corner or a whole edge of both of them. For simplicity, we denote by α_k the constant value of α on each Ω_k, $1 \leq k \leq K$.
We make the further (and non restrictive) assumption that the intersection of each $\partial\Omega_k$ with $\partial\Omega$ is a corner or a whole edge of Ω_k. Thus, the skeleton S of the decomposition, equal to $\bigcup_{k=1}^{K} \partial\Omega_k \backslash \partial\Omega$, admits a decomposition without overlapping into mortars

$$\overline{S} = \bigcup_{m=1}^{M} \overline{\gamma}_m \text{ tel que } \gamma_m \cap \gamma_{m'} = \emptyset,\ \text{ pour } m \neq m', \tag{11}$$

where each γ_m is a whole edge of one of the Ω_k, which is then denoted by $\Omega_{k(m)}$. Note that the choice of this decomposition is not unique, however it is decided a priori for all the discretizations we work with.
In order to describe the discrete problem, we introduce the discretization parameter δ, which is here a K-tuple of positive integers N_k, $1 \leq k \leq K$. Indeed, the local discrete space on each Ω_k is the space $\mathbb{P}_{N_k}(\Omega_k)$ of restrictions to Ω_k of polynomials with degree $\leq N_k$ with respect to each variable. In all that follows, c stands for a generic constant which may vary from one line to the other but is always independent of δ.
The $\Gamma^{k,j}$, $1 \leq j \leq 4$ are the corners of Ω_k, $1 \leq k \leq K$.
We now introduce the discrete spaces. For each k, $1 \leq k \leq K$, the discrete space of velocities $\mathbb{X}_\delta$ is defined by

$$\mathbb{X}_\delta = \left\{ \boldsymbol{v}_\delta \in L^2(\Omega)^2; \boldsymbol{v}_{\delta|\Omega_k} \in \mathbb{P}_{N_k}(\Omega_k)^2, 1 \leq k \leq K \right\}. \tag{12}$$

According to the standard mortar element approach [11, Sec. 2] and [10], we associate with each piecewise regular function q its mortar function $\Phi_m(q)$: On each γ_m, $1 \leq m \leq M$, the restriction of $\Phi_m(q)$ to γ_m is equal to the trace of $q_{|\Omega_{k(m)}}$. The discrete space of pressures is the space $\mathbb{M}_\delta$ of functions q_δ
(i) which belong to $L_0^2(\Omega)$,
(ii) such that their restriction to each Ω_k, $1 \leq k \leq K$, belongs to $\mathbb{P}_{N_k}(\Omega_k)$,
(iii) such that the following matching condition holds on all subdomains Ω_k, $1 \leq k \leq K$, and for all edges $\Gamma^{k,j}$ of Ω_k that are not contained in $\partial\Omega$,

$$\forall \varphi \in \mathbb{P}_{N_k-2}(\Gamma^{k,j}),\ \int_{\Gamma^{k,j}} (q_{\delta|\Omega_k} - \Phi(q_\delta))(\tau)\varphi(\tau)\, d\tau = 0, \tag{13}$$

where $\mathbb{P}_{N_k-2}(\Gamma^{k,j})$ is the space of polynomials with degree $\leq N_k - 2$ on $\Gamma^{k,j}$, and τ denotes the tangential coordinate on $\Gamma^{k,j}$. Note that the quantity $q_{\delta|\Omega_k} - \Phi(q_\delta)$ represents the jump of q_δ through $\Gamma^{k,j}$, where $\Gamma^{k,j}$ is not one of the γ_m.
Note that condition (13) is obviously satisfied on all $\Gamma^{k,j}$ which coincide with a γ_m and also that, except for some rather special decomposition, the space $\mathbb{M}_\delta$ is not contained in $H^1(\Omega)$, which means that the discretization is not conforming.
We recall the Gauss-Lobatto formula on the interval $]-1,1[$: for each positive integer N, with the notation $\xi_0^N = -1$ and $\xi_N^N = 1$, there exists a unique set of nodes ξ_j^N, $1 \leq j \leq N-1$, and weights ρ_j, $0 \leq j \leq N$, such that

$$\forall \Phi \in \mathbb{P}_{2N-1}(-1,1), \int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j^N)\rho_j^N. \tag{14}$$

The ξ_j^N are equal to the zeros of the first derivative of the Legendre polynomial of the degree N and the ρ_j^N are positive. Moreover, the following positivity property holds (sse [13])

$$\forall \varphi_N \in \mathbb{P}_N(-1,1), \;\; \|\varphi_N\|^2_{L^2(-1,1)} \leq \sum_{j=0}^{N} \varphi_N^2(\xi_j^N)\rho_j^N \leq 3\|\varphi_N\|^2_{L^2(-1,1)}. \tag{15}$$

Next, on each Ω_k, we take N equal to N_k and, by homothety and translation, we construct from the $\xi_j^{N_k}$ and $\rho_j^{N_k}$, $0 \leq j \leq N_k$, the nodes and the weights $\xi_{kj}^{(x)}$ and $\rho_{kj}^{(x)}$, resp. $\xi_{kj}^{(y)}$ and $\rho_{kj}^{(y)}$, in the x-direction, resp. in the y-direction (the exponent N_k is omitted for simplicity). This leads to a discrete product on all functions $\boldsymbol{u}$ and $\boldsymbol{v}$ which have continuous restrictions to all $\overline{\Omega}_k$, $1 \leq k \leq K$:

$$((\boldsymbol{u},\boldsymbol{v}))_\delta = \sum_{k=1}^{K} ((\boldsymbol{u},\boldsymbol{v}))^k_{N_k}, \tag{16}$$

with

$$((\boldsymbol{u},\boldsymbol{v}))^k_{N_k} = \sum_{i=0}^{N_k}\sum_{j=0}^{N_k} \boldsymbol{u}(\xi_{ki}^{(x)},\xi_{kj}^{(y)})\boldsymbol{v}(\xi_{ki}^{(x)},\xi_{kj}^{(y)})\rho_{ki}^{(x)}\rho_{kj}^{(y)}.$$

It follows from the exactness property (14) that the product $((.,.))_\delta$ coincides with the scalar product of $L^2(\Omega)$ whenever the restriction of the product $\boldsymbol{uv}$ to all Ω_k belong to $\mathbb{P}_{2N_k-1}(\Omega_k)$.
Also, we defined the global scalar product on $\partial\Omega$

$$((\boldsymbol{u}_\delta,\boldsymbol{v}_\delta))^{\partial\Omega}_\delta = \sum_{\{\Gamma^{k,j}\subset\partial\Omega\}} (\boldsymbol{u}_\delta,\boldsymbol{v}_\delta)^{\Gamma^{k,j}}_{N_k}, \tag{17}$$

where

$$(\boldsymbol{u}_\delta,\boldsymbol{v}_\delta)^{\partial\Omega}_N = \sum_{j=1}^{2d}\sum_{\boldsymbol{x}\in\Xi_N\cap\overline{\Gamma}_j} \boldsymbol{u}_\delta(\boldsymbol{x})\boldsymbol{v}_\delta(\boldsymbol{x})\rho_{\boldsymbol{x}}, \tag{18}$$

We assume that the functions f and g has continuous restrictions to all $\overline{\Omega}_k$, $1 \leq k \leq K$ and $\partial\Omega$ respectively. Then, the discrete problem reads :

Find $(\boldsymbol{u}_\delta, p_\delta) \in \mathbb{X}_\delta \times \mathbb{M}_\delta$ such that

$$\left\{\begin{array}{lcl} \forall \boldsymbol{v}_\delta \in \mathbb{X}_\delta,\ a_\alpha^\delta(\boldsymbol{u}_\delta, \boldsymbol{v}_\delta) + b_\delta(\boldsymbol{v}_\delta, p_\delta) & = & ((\alpha \boldsymbol{f}, \boldsymbol{v}_\delta))_\delta, \\ \forall q_\delta \in \mathbb{M}_\delta,\ b_\delta(\boldsymbol{u}_\delta, q_\delta) & = & ((g, q_\delta))_\delta^{\partial\Omega}, \end{array}\right. \tag{19}$$

where the bilinears forms $a_\alpha^\delta(.,.)$ and b_δ are defined by

$$a_\alpha^\delta(\boldsymbol{u}_\delta, \boldsymbol{v}_\delta) = \sum_{k=1}^K \alpha_k ((\boldsymbol{u}_\delta, \boldsymbol{v}_\delta))_{N_k}^k, \quad b_\delta(\boldsymbol{v}_\delta, q_\delta) = \sum_{k=1}^K ((\boldsymbol{v}_\delta, \mathbf{grad}\, q_\delta))_{N_k}^k. \tag{20}$$

Several steps are needed for proving the well-posedness of this problem.

Lemma 4 *The form $a_\alpha^\delta(.,.)$ satisfies the following properties of continuity*

$$\forall \boldsymbol{u}_\delta \in \mathbb{X}_\delta,\ \forall \boldsymbol{v}_\delta \in \mathbb{X}_\delta,\ a_\alpha^\delta(\boldsymbol{u}_\delta, \boldsymbol{v}_\delta) \leq 9 \|\boldsymbol{u}_\delta\|_\alpha \|\boldsymbol{v}_\delta\|_\alpha, \tag{21}$$

and of ellipticity

$$\forall \boldsymbol{u}_\delta \in \mathbb{X}_\delta,\ a_\alpha^\delta(\boldsymbol{u}_\delta, \boldsymbol{u}_\delta) \geq \|\boldsymbol{u}_\delta\|_\alpha^2. \tag{22}$$

Proof. Thanks to a double Cauchy-Schwarz inequality, we have

$$a_\alpha^\delta(\boldsymbol{u}_\delta, \boldsymbol{v}_\delta) \leq a_\alpha^\delta(\boldsymbol{u}_\delta, \boldsymbol{u}_\delta)^{\frac{1}{2}} a_\alpha^\delta(\boldsymbol{v}_\delta, \boldsymbol{v}_\delta)^{\frac{1}{2}},$$

so that it suffices to bound $a_\alpha^\delta(\boldsymbol{u}_\delta, \boldsymbol{u}_\delta)$. Thanks to the positivity property (15), we have

$$\sum_{k=1}^K \alpha_k \|\boldsymbol{u}_\delta\|_{L^2(\Omega_k)^2}^2 \leq a_\alpha^\delta(\boldsymbol{u}_\delta, \boldsymbol{u}_\delta) \leq \sum_{k=1}^K 9\alpha_k \|\boldsymbol{u}_\delta\|_{L^2(\Omega_k)^2}^2.$$

So, the desired results. ■

Since $\mathbb{M}_\delta$ is not contained in $H^1(\Omega)$, we prove that the "broken" energy norm defined by

$$\|q\|_{\alpha*} = \left(\sum_{k=1}^K \alpha_k^{-1} |q|_{H^1(\Omega_k)}^2 \right)^{\frac{1}{2}}, \tag{23}$$

is still a norm on $\mathbb{M}_\delta$.

Lemma 5 *The quantity $\|.\|_{\alpha*}$ defined in* (23) *is a norm on $\mathbb{M}_\delta$. Moreover, there exist a constant C independent of δ such that the following property holds :*

$$\forall q \in \mathbb{N}(\Omega) \cap L_0^2(\Omega),\ \sum_{k=1}^K \|q\|_{L^2(\Omega_k)}^2 \leq C\sqrt{\alpha_{\max}} \|q\|_{\alpha^*}^2. \tag{24}$$

We suppose that $N_K \geq N_D - 2$, when N_D denote the maximal number of the set of all vertices of Ω_k that are inside en edge of another subdomains.

For the proof see [10].

From now on, we work with the norm $\|.\|_{\alpha*}$, and we suppose that $N_K \geq N_D - 2$ is checked. The following continuity property is obvious :

$$\forall \boldsymbol{v}_\delta \in \mathbb{X}_\delta, \forall q_\delta \in \mathbb{M}_\delta, \quad b_\delta(\boldsymbol{v}_\delta, q_\delta) \leq \|\boldsymbol{v}_\delta\|_\alpha \|q_\delta\|_{\alpha^*}. \tag{25}$$

Moreover, we note that, for any q_δ in $\mathbb{M}_\delta$, the function $\boldsymbol{v}_\delta$ defined by

$$\boldsymbol{v}_{\delta|\Omega_k} = \alpha_k^{-1} \mathbf{grad}\,(q_{\delta|\Omega_k}),\ \forall 1 \leq k \leq K, \tag{26}$$

belongs to $\mathbb{X}_\delta$. So, the following inf-sup condition is derived by taking $\boldsymbol{v}_\delta$ as in (26).

Lemma 6 *The form $b_\delta(,)$ satisfies the inf-sup condition*

$$\forall q_\delta \in \mathbb{M}_\delta,\ \sup_{\boldsymbol{v}_\delta \in \mathbb{X}_\delta} \frac{b_\delta(\boldsymbol{v}_\delta, q_\delta)}{\|\boldsymbol{v}_\delta\|_\alpha} \geq \|q_\delta\|_{\alpha*}. \tag{27}$$

We introduce the Lagrange interpolation operator $\mathcal{I}_\delta^k$, $1 \leq k \leq K$, operator on all nodes $(\xi_{ki}^{(x)}, \xi_{kj}^{(y)})$, $0 \leq i, j \leq N_k$, with values in $\mathbb{P}_{N_k}(\Omega_k)$, and finally the global operator $\mathcal{I}_\delta$ by

$$(\mathcal{I}_\delta \boldsymbol{v})_{|\Omega_k} = \mathcal{I}_\delta^k \boldsymbol{v}_{|\Omega_k}, \quad 1 \leq k \leq K. \tag{28}$$

We are now in position to prove the well-posedness of problem (5).

Theorem 7 *For any data $(\boldsymbol{f}, g)$ such that each $\boldsymbol{f}_{|\Omega_k}$, $1 \leq k \leq K$, and g are continuous on $\overline{\Omega}_k$ and on $\partial\Omega$ respectively, problem (19) has a unique solution $(\boldsymbol{u}_\delta, p_\delta)$ in $\mathbb{X}_\delta \times \mathbb{M}_\delta$. Moreover, there exists a constant c independent of δ suvh that this solution satisfies*

$$\|\boldsymbol{u}_\delta\|_\alpha + \|p_\delta\|_{\alpha*} \leq c\,\sqrt{\alpha_{\max}}(\|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^2} + \|\mathcal{I}_\delta^{\partial\Omega} g\|_{H^{-\frac{1}{2}}(\partial\Omega)}), \tag{29}$$

Proof. We establish successively the existence and uniqueness of the solution.
1) It follows from the Lax-Milgram lemma, combined with Bramble-Hilbert inequality and the lemma 5, that there exists a unique φ_δ in $\mathbb{M}_\delta$ such that

$$\forall \psi_\delta \in \mathbb{M}_\delta,\ ((\mathbf{grad}\,\varphi_\delta, \mathbf{grad}\,\psi_\delta))_\delta = ((g, \psi_\delta))_\delta^{\partial\Omega},$$

Thus, the function $\boldsymbol{u}_\delta^b = \mathbf{grad}\,\varphi_\delta$, satisfies

$$\|\boldsymbol{u}_\delta^b\|_\alpha \leq c\|\mathcal{I}_\delta^{\partial\Omega} g\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{30}$$

On the other hand, it follows for the standard results on saddle-point problems, see [15, Chap. I, Cor. 4.1], combined with (22), (27) and the inf-sup condition (27), that the problem

Find $(\boldsymbol{u}_\delta^0, p_\delta) \in \mathbb{X}_\delta \times \mathbb{M}_\delta$ such that

$$\left\{\begin{array}{llcl} \forall \boldsymbol{v}_\delta \in \mathbb{X}_\delta, & a_\alpha^\delta(\boldsymbol{u}_\delta^0, \boldsymbol{v}_\delta) + b_\delta(\boldsymbol{v}_\delta, p_\delta) & = & \sum_{k=1}^{K^*} \alpha_k((\boldsymbol{f}, \boldsymbol{v}_\delta))_\delta - a_\alpha^\delta(\boldsymbol{u}_\delta^b, \boldsymbol{v}_\delta), \\ \forall q_\delta \in \mathbb{M}_\delta, & b_\delta(\boldsymbol{u}_\delta^0, q_\delta) & = & 0, \end{array}\right. \tag{31}$$

has a unique solution $(\boldsymbol{u}_\delta^0, p_\delta)$ which moreover satisfies

$$\|\boldsymbol{u}_\delta^0\|_\alpha + \|p_\delta\|_{\alpha^*} \le c\sqrt{\alpha_{\max}}(\|\mathcal{I}_\delta \boldsymbol{f}\|_{L^2(\Omega)^d} + \|\boldsymbol{u}_\delta^b\|_\alpha). \tag{32}$$

Then, the pair $(\boldsymbol{u}_\delta, p_\delta)$, with $\boldsymbol{u}_\delta = \boldsymbol{u}_\delta^0 + \boldsymbol{u}_\delta^b$, is a solution of problem (19), and estimate (29) follows from (30) and (32).
2) Let $(\boldsymbol{u}_\delta, p_\delta)$ be a solution of problem (19) with data $(\boldsymbol{f}, g)$ equal to zero. Taking $\boldsymbol{v}_\delta$ equal to $\boldsymbol{u}_\delta$ in (19) and using (22) yields that $\boldsymbol{u}_\delta$ is zero. Then, the fact that p_δ is zero follows from (27). This proves the uniqueness of the solution $(\boldsymbol{u}_\delta, p_\delta)$. ■

To conclude, we introduce the discrete kernel

$$V_\delta = \{\boldsymbol{v}_\delta \in \mathbb{X}_\delta; \forall q_\delta \in \mathbb{M}_\delta, b_\delta(\boldsymbol{v}_\delta, q_\delta) = 0\}. \tag{33}$$

As usual, it plays a key role in the numerical analysis of problem (19).

4 Error estimates

This section is devoted to the proof of an error estimate, first for the velocity, second for the pressure. We intend to prove an error estimate between the solution $(\boldsymbol{u}, p)$ of problem (5) and the solution $(\boldsymbol{u}_\delta, p_\delta)$ of problem (19). So we announce the following theorem and we describe their proof.

Theorem 8 *Assume that the function α is constant on each Ω_k, $1 \le k \le K$. If the solution $(\boldsymbol{u}, p)$ of problem (5) is such its restriction to each Ω_k, $1 \le k \le K$, belongs to $H^{s_k}(\Omega_k)^2 \times H^{s_k+1}(\Omega_k)$, $s_k \ge \frac{1}{2}$, and if the function $\boldsymbol{f}$ is such that its restriction to each Ω_k, $1 \le k \le K$, belongs to $H^{\sigma_k}(\Omega_k)$, $\sigma_k > 1$, the following error estimate holds between this solution $(\boldsymbol{u}, p)$ and the solution $(\boldsymbol{u}_\delta, p_\delta)$ of problem (19)*

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_\delta\|_\alpha & \\ \le c\Bigg(& (1+\mu+\mu_\delta)^{\frac{1}{2}} \sum_{k=1}^K N_k^{-s_k}\Big(\alpha_k^{\frac{1}{2}}(\log N_k)\|\boldsymbol{u}_{|\Omega_k}\|_{H^{s_k}(\Omega_k)^2} + \alpha_k^{-\frac{1}{2}}\|p_{|\Omega_k}\|_{H^{s_k+1}(\Omega_k)}\Big) \\ & + \sqrt{\alpha_{\max}}\Big(\sum_{k=1}^K N_k^{-2\sigma_k}\|\boldsymbol{f}_{|\Omega_k}\|^2_{H^{\sigma_k}(\Omega_k)^2}\Big)^{1/2}\Bigg)\end{aligned} \tag{34}$$

where the constant c is independent of the parameter δ and the function α.

Proof. For a proof we need a several steps and lemmas. Let $\boldsymbol{w}_\delta$ be any function in the kernel V_δ. Multiplying the first line of (19) by $\boldsymbol{w}_\delta$ gives

$$\sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{u}_0 \cdot \boldsymbol{w}_\delta \, d\boldsymbol{x} + b(\boldsymbol{w}_\delta, p) = \sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{f} \cdot \boldsymbol{w}_\delta \, d\boldsymbol{x} - a_\alpha(\boldsymbol{u}_b, \boldsymbol{w}_\delta),$$

using the definition of V_δ thus implies, for any q_δ in $\mathbb{M}_\delta$,

$$\sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{u}_0 \cdot \boldsymbol{w}_\delta \, d\boldsymbol{x} + b(\boldsymbol{w}_\delta, p - q_\delta) = \sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{f} \cdot \boldsymbol{w}_\delta \, d\boldsymbol{x} - a_\alpha(\boldsymbol{u}_b, \boldsymbol{w}_\delta). \tag{35}$$

So, we deduce from ellipticity property (22), that we have for any $\boldsymbol{v}_\delta$ in V_δ

$$\|\boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta\|_\alpha^2 \quad \leq \quad a_\alpha^\delta(\boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta, \boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta).$$

Adding (35) with $\boldsymbol{w}_\delta = \boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta$ and subtracting the first line of (19) leads to

$$\begin{aligned}
\|\boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta\|_\alpha^2 \leq &\sum_{k=1}^K \alpha_k \int_{\Omega_k} (\boldsymbol{u}_0 - \boldsymbol{v}_\delta)(\boldsymbol{x}) \cdot (\boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta)(\boldsymbol{x})\, d\boldsymbol{x} \\
&+ \sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{v}_\delta(\boldsymbol{x}) \cdot (\boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta)(\boldsymbol{x})\, d\boldsymbol{x} - ((\alpha \boldsymbol{v}_\delta, \boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta))_\delta \\
&+ \int_\Omega (\boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta)(\boldsymbol{x}) \cdot \mathbf{grad}\,(p - q_\delta)(\boldsymbol{x})\, d\boldsymbol{x} \\
&+ ((\alpha \boldsymbol{f}, \boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta))_\delta - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{f}(\boldsymbol{x}) \cdot (\boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta)(\boldsymbol{x})\, d\boldsymbol{x}, \\
&- a_\delta(\boldsymbol{u}_\delta^b, \boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta) + a_\alpha(\boldsymbol{u}_b, \boldsymbol{u}_\delta^0 - \boldsymbol{v}_\delta).
\end{aligned}$$

By combining property of continuity (25) and triangle inequality, we derive that the error $\|\boldsymbol{u} - \boldsymbol{u}_\delta\|_\alpha$ is bounded, up to a multiplicative constant, by the sum of five terms :

- the approximation error in $\mathbb{X}_\delta$

$$\inf_{\boldsymbol{v}_\delta \in V_\delta} \|\boldsymbol{u}_0 - \boldsymbol{v}_\delta\|_\alpha, \tag{36}$$

- the error approximation in $\mathbb{M}_\delta$

$$\inf_{q_\delta \in \mathbb{M}_\delta} \|p - q_\delta\|_{\alpha_*}, \tag{37}$$

- three terms issued from numerical integration

$$\sup_{\boldsymbol{w}_\delta \in \mathbb{X}_\delta} \frac{(a_\alpha - a_\alpha^\delta)(\boldsymbol{v}_\delta, \boldsymbol{w}_\delta)}{\|\boldsymbol{w}_\delta\|_\alpha}, \tag{38}$$

$$\sup_{\boldsymbol{w}_\delta \in \mathbb{X}_\delta} \frac{a_\alpha(\boldsymbol{u}_b, \boldsymbol{w}_\delta) - a_\alpha^\delta(\boldsymbol{u}_\delta^b, \boldsymbol{w}_\delta)}{\|\boldsymbol{w}_\delta\|_\alpha}, \tag{39}$$

and

$$\sup_{\boldsymbol{w}_\delta \in \mathbb{X}_\delta} \frac{((\alpha \boldsymbol{f}, \mathbf{w}_\delta))_\alpha - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{w}_\delta(\boldsymbol{x})\, d\boldsymbol{x}}{\|\boldsymbol{w}_\delta\|_\alpha}. \tag{40}$$

Estimating the terms issued from numerical integration is easy since they can be evaluated separately on each Ω_k. For each k, $1 \leq k \leq K$, let Π_{N_k-1} denote the orthogonal projection operator from $L^2(\Omega_k)$ onto $\mathbb{P}_{N_k-1}(\Omega_k)$. For any $\boldsymbol{w}_\delta$ in $\mathbb{X}_\delta$, since each product of $\Pi_{N_k-1}\boldsymbol{u}$ by $\boldsymbol{w}_\delta$ belongs to $\mathbb{P}_{2N_k-1}(\Omega)$, it follows from the exactness property (14) that

$$(a_\alpha - a_\alpha^\delta)(\boldsymbol{v}_\delta, \boldsymbol{w}_\delta) = \sum_{k=1}^K \alpha_k \Big(\int_{\Omega_k} (\boldsymbol{v}_\delta - \Pi_{N_k-1}\boldsymbol{u}_0) \cdot \boldsymbol{w}_\delta\, d\boldsymbol{x} - ((\boldsymbol{v}_\delta - \Pi_{N_k-1}\boldsymbol{u}_0, \boldsymbol{w}_\delta))_{N_k}^k \Big).$$

So, we deduce from the continuity property (21) that

$$\begin{aligned}\sup_{\boldsymbol{w}_\delta\in\mathbb{X}_\delta}\frac{(a_\alpha-a_\alpha^\delta)(\boldsymbol{v}_\delta,\boldsymbol{w}_\delta)}{\|\boldsymbol{w}_\delta\|_\alpha} &\leq 10\left(\sum_{k=1}^K\alpha_k\|\boldsymbol{v}_\delta-\Pi_{N_k-1}\boldsymbol{u}_0\|^2_{L^2(\Omega_k)^d}\right)^{\frac12}\\ &\leq 10\|\boldsymbol{u}_0-\boldsymbol{v}_\delta\|_\alpha+10\left(\sum_{k=1}^K\alpha_k\|\boldsymbol{u}_0-\Pi_{N_k-1}\boldsymbol{u}_0\|^2_{L^2(\Omega_k)^d}\right)^{\frac12}.\end{aligned}$$

The approximation properties of the operator Π_{N_k-1} are well known (see, for example, Theorem 7.3 of [7] and Proposition 2.6 of [8]), they lead to the following estimate : if the solution $\boldsymbol{u}_0$ is such that each $\boldsymbol{u}_{0|\Omega_k}$ belongs to $H^{s_k+1}(\Omega_k)^2$, $s_k\geq 0$

$$\sup_{\boldsymbol{w}_\delta\in\mathbb{X}_\delta}\frac{(a_\alpha-a_\alpha^\delta)(\boldsymbol{v}_\delta,\boldsymbol{w}_\delta)}{\|\boldsymbol{w}_\delta\|_\alpha}\leq 4\|\boldsymbol{u}_0-\boldsymbol{v}_\delta\|_\alpha+c\left(\sum_{k=1}^K\alpha_k N_k^{-2s_k}\|\boldsymbol{u}_{0|\Omega_k}\|^2_{H^{s_k}(\Omega_k)^2}\right)^{\frac12}. \tag{41}$$

Similarly, for any $\boldsymbol{w}_\delta$ in $\mathbb{X}_\delta$, we have

$$\begin{aligned}&((\alpha\boldsymbol{f},\boldsymbol{w}_\delta))_\delta-\sum_{k=1}^K\alpha_k\int_{\Omega_k}\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{w}_\delta(\boldsymbol{x})\,d\boldsymbol{x}\\ &=\sum_{k=1}^K\alpha_k\left(((\mathcal{I}_\delta\boldsymbol{f}-\Pi_{N_k-1}\boldsymbol{f},\boldsymbol{w}_\delta))^k_{N_k}-\int_{\Omega_k}(\boldsymbol{f}-\Pi_{N_k-1}\boldsymbol{f})(\boldsymbol{x})\cdot\boldsymbol{w}_\delta(\boldsymbol{x})\,d\boldsymbol{x}\right).\end{aligned}$$

So, using (14) yields

$$\begin{aligned}&((\alpha\boldsymbol{f},\boldsymbol{w}_\delta))_\delta-\sum_{k=1}^K\alpha_k\int_{\Omega_k}\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{w}_\delta(\boldsymbol{x})\,d\boldsymbol{x}\\ &\leq\sqrt{\alpha_{\max}}\left(10\left(\sum_{k=1}^K\|\boldsymbol{f}-\Pi_{N_k-1}\boldsymbol{f}\|^2_{L^2(\Omega_k)^2}\right)^{\frac12}+9\|\boldsymbol{f}-\mathcal{I}_\delta\boldsymbol{f}\|_{L^2(\Omega_k)^2}\right)\|\boldsymbol{w}_\delta\|_{L^2(\Omega)^2}.\end{aligned}$$

To bound $\|\boldsymbol{w}_\delta\|_{L^2(\Omega)^2}$ as a function of $\|\boldsymbol{w}_\delta\|_\alpha$ and the approximation properties of the operators $\mathcal{I}_\delta$ et Π_{N_k-1} (Theorem 7.1 of [7] and Theorem 14.2 of [8]), we derive that, if the function $\boldsymbol{f}$ is such that each $\boldsymbol{f}_{|\Omega_k}$ belongs to $H^{\sigma_k}(\Omega_k)^2$, $\sigma_k>1$,

$$\sup_{\boldsymbol{w}_\delta\in\mathbb{X}_\delta}\frac{((\alpha\boldsymbol{f},\boldsymbol{w}_\delta))_\delta-\sum_{k=1}^K\alpha_k\int_{\Omega_k}\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{w}_\delta(\boldsymbol{x})\,d\boldsymbol{x}}{\|\boldsymbol{w}_\delta\|_\alpha}\leq c\sqrt{\alpha_{\max}}\left(\sum_{k=1}^K N_k^{-2\sigma_k}\|\boldsymbol{f}_{|\Omega_k}\|^2_{H^{\sigma_k}(\Omega_k)^2}\right)^{\frac12}. \tag{42}$$

By analogy, we estimate the term (39), and by (30), we have

$$\sup_{\boldsymbol{w}_\delta\in\mathbb{X}_\delta}\frac{a_\alpha(\boldsymbol{u}_b,\boldsymbol{w}_\delta)-a_\alpha^\delta(\boldsymbol{u}_\delta^b,\boldsymbol{w}_\delta)}{\|\boldsymbol{w}_\delta\|_\alpha}\leq c\sqrt{\alpha_{\max}}\left(\sum_{k=1}^K N_k^{-2s_k}\|\boldsymbol{u}_{b|\Omega_k}\|^2_{H^{s_k}(\Omega_k)^2}\right)^{\frac12}. \tag{43}$$

To estimate the term (36), we need a lemma.

Lemma 9 *There exists a constant c independent of δ such that*

$$\inf_{\boldsymbol{v}_\delta \in V_\delta} \|\boldsymbol{u}_0 - \boldsymbol{v}_\delta\|_\alpha \le c \left(\inf_{\boldsymbol{z}_\delta \in \mathbb{X}_\delta} \|\boldsymbol{u}_0 - \boldsymbol{z}_\delta\|_\alpha + \sup_{q_\delta \in \mathbb{M}_\delta} \frac{\int_S (\boldsymbol{u}_0 \cdot \boldsymbol{n})[q_\delta]\, d\boldsymbol{\tau}}{\|q_\delta\|_{\alpha*}} \right). \tag{44}$$

Proof. Let $\boldsymbol{z}_\delta$ be an arbitrary element of $\mathbb{X}_\delta$. The inf-sup condition (27) and [15] prove there exists a unique $\boldsymbol{t}_\delta \in V_\delta^\perp$ such that

$$b_\delta(\boldsymbol{t}_\delta, q_\delta) = b_\delta(\boldsymbol{z}_\delta, q_\delta) \quad \text{and } \|\boldsymbol{t}_\delta\|_\alpha \le \frac{1}{\beta} \sup_{q_\delta \in \mathbb{M}_\delta} \frac{b_\delta(\boldsymbol{z}_\delta, q_\delta)}{\|q_\delta\|_{\alpha*}}.$$

Thus, if we set $\boldsymbol{v}_\delta = \boldsymbol{z}_\delta - \boldsymbol{t}_\delta$, then by combining the exactness property (14) and the integration by parties, we have

$$b_\delta(\boldsymbol{u}_0, q_\delta) = \sum_{k=1}^K \int_{\Omega_k} \boldsymbol{u}_0 \cdot \mathbf{grad}\, q_\delta\, d\boldsymbol{x} = \sum_{k=1}^K \int_{\partial\Omega_k} (\boldsymbol{u}_0 \cdot \boldsymbol{n}) q_\delta\, d\boldsymbol{\tau} = \int_S (\boldsymbol{u}_0 \cdot \boldsymbol{n})[q_\delta]\, d\boldsymbol{\tau},$$

therefore

$$\|\boldsymbol{t}_\delta\|_\alpha \le C \left(\sup_{q_\delta \in \mathbb{M}_\delta} \frac{b_\delta(\boldsymbol{u}_0 - \boldsymbol{z}_\delta, q_\delta) - \int_S (\boldsymbol{u}_0 \cdot \boldsymbol{n})[q_\delta]\, d\boldsymbol{\tau}}{\|q_\delta\|_{\alpha*}} \right).$$

This inequality and triangle inequality implies

$$\begin{aligned} \|\boldsymbol{u}_0 - \boldsymbol{v}_\delta\|_\alpha &\le \|\boldsymbol{u}_0 - \boldsymbol{z}_\delta\|_\alpha + \|\boldsymbol{t}_\delta\|_\alpha \\ &\le c \left(\|\boldsymbol{u}_0 - \boldsymbol{z}_\delta\|_\alpha + \sup_{q_\delta \in \mathbb{M}_\delta} \frac{\int_S (\boldsymbol{u}_0 \cdot \boldsymbol{n})[q_\delta]\, d\boldsymbol{\tau}}{\|q_\delta\|_{\alpha*}} \right). \end{aligned}$$

As $\boldsymbol{z}_\delta$ is arbitrary, this implies (44). ■

Estimating the approximation error in $\mathbb{X}_\delta$ is derived simply by taking $\boldsymbol{w}_\delta$ equal to the orthogonal projection operator $\Pi_{N_k-1}\boldsymbol{u}_0$ on each Ω_k.

Lemma 10 *Assume that the solution* $(\boldsymbol{u}, p)$ *of problem* (5) *is such that each* $\boldsymbol{u}_{|\Omega_k}$ *belongs to* $H^{s_k}(\Omega_k)^2$ *for a real number* s_k, $s_k \geqslant 0$. *The following estimate holds*

$$\inf_{\boldsymbol{z}_\delta \in \mathbb{X}_\delta} \|\boldsymbol{u}_0 - \boldsymbol{z}_\delta\|_\alpha \le c\Big(\sum_{k=1}^K \alpha_k N_k^{-2s_k} \|\boldsymbol{u}_{|\Omega_k}\|^2_{H^{s_k}(\Omega_k)^2} \Big)^{\frac12}. \tag{45}$$

Now, we evaluate the consistency error. It involves the quantity μ, defined as the largest ratio

$$\mu = \max_{1 \le m \le M} \max_{\ell \in \mathcal{E}(m)} \left(\frac{\alpha_\ell^{-1}}{\alpha_{k(m)}^{-1}} \right)^{\frac12}, \tag{46}$$

where, for each m, $1 \le m \le M$, $\mathcal{E}(m)$ is the set of indices k, $1 \le k \le K$, such that $\partial\Omega_k \cap \gamma_m$ has a positive measure. Note that this constant depends on the decomposition and on the choice of the mortars but not on the discretization parameter.

In order to evaluate the consistency error, we introduce the orthogonal projection operator $\pi_{N_k-2}^{\Gamma^{k,j}}$ from $L^2(\Gamma^{k,j})$ onto $\mathbb{P}_{N_k-2}(\Gamma^{k,j})$. We recall the following properties of this operator (see [11]) : For any nonnegative real numbers s and t, and for any function φ in $H^s(\Gamma^{k,j})$,

$$\|\varphi - \pi_{N_k-2}^{\Gamma^{k,j}} \varphi\|_{H^{-t}(\Gamma^{k,j})} \le c N_k^{s+t} \|\varphi\|_{H^s(\Gamma^{k,j})}. \tag{47}$$

Lemma 11 *Assume that the solution* $(\boldsymbol{u}, p)$ *of problem* (5) *is such that each* $\boldsymbol{u}_{|\Omega_k}$, $1 \le k \le K$, *belongs to* $H^{s_k}(\Omega_k)^2$ *for a real number* s_k, $s_k \ge \frac{1}{2}$, *the following estimate holds*

$$\sup_{q_\delta \in \mathbb{M}_\delta} \frac{\int_S (\boldsymbol{u}_0 \cdot \boldsymbol{n})[q_\delta]\, d\boldsymbol{\tau}}{\|q_\delta\|_{\alpha *}} \le c(1+\mu) \left(\sum_{k=1}^{K} \alpha_k N_k^{-2s_k} (\log N_k) \|\boldsymbol{u}_{|\Omega_k}\|^2_{H^{s_k}(\Omega_k)^d} \right)^{1/2}. \tag{48}$$

Remark 12 *In fact, the* $(logN_k)^{\frac{1}{2}}$ *in* (48) *disappears when all the edges of* $\partial\Omega_k$ *which are not mortars are contained either in* $\partial\Omega$ *or in one mortar, however it is negligible in comparison with the* $N_k^{s_k}$ *when* N_k *is large enough.*

Estimating the approximation error in $\mathbb{M}_\delta$ is more complex. See [11] for the proof.

The lemma gives a bound for the approximation error. Here, we introduce the quantity

$$\mu_\delta = \max_{1 \le m \le M} \max_{\ell \in \mathcal{E}(m)} \left(\frac{\alpha_\ell^{-1} N_\ell^{-1}}{\alpha_{k(m)}^{-1} N_{k(m)}^{-1}} \right)^{\frac{1}{2}}, \tag{49}$$

which now depends on δ.

Lemma 13 *Assume that the solution* $(\boldsymbol{u}, p)$ *of problem* (5) *is such that each* $p_{|\Omega_k}$, $1 \le k \le K$, *belongs to* $H^{s_k+1}(\Omega_k)$ *for a real number* s_k, $s_k > 0$. *The following estimate holds*

$$\inf_{q_\delta \in \mathbb{M}_\delta} \|p - q_\delta\|_{\alpha *} \le c(1+\mu+\mu_\delta) \left(\sum_{k=1}^{K} \alpha_k^{-\frac{1}{2}} N_k^{-2s_k} \|p_{|\Omega_k}\|^2_{H^{s_k+1}(\Omega_k)} \right)^{\frac{1}{2}}. \tag{50}$$

For the proof we refer to [12]. ■

From the previous remarks, and the reference [5], the following improved estimate holds for a conforming decomposition.

Corollary 14 *If the decomposition* (10) *is conforming and if the assumptions of Theorem 8 are satisfied, the following error estimate holds between the solution* $(\boldsymbol{u}, p)$ *of problem* (5) *and the solution* $(\boldsymbol{u}_\delta, p_\delta)$ *of problem* (19)

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_\delta\|_\alpha \\ &\quad \le c\Bigg((1+\mu)^{\frac{1}{2}} \sum_{k=1}^{K} N_k^{-s_k} \Big(\alpha_k^{\frac{1}{2}} \|\boldsymbol{u}_{|\Omega_k}\|_{H^{s_k}(\Omega_k)^2} + \alpha_k^{-\frac{1}{2}} \|p_{|\Omega_k}\|_{H^{s_k+1}(\Omega_k)} \Big) \\ &\qquad\qquad + \sqrt{\alpha_{\max}} \Big(\sum_{k=1}^{K} N_k^{-2\sigma_k} \|\boldsymbol{f}_{|\Omega_k}\|^2_{H^{\sigma_k}(\Omega_k)^2} \Big)^{1/2} \Bigg), \end{aligned} \tag{51}$$

where the constant c *is independent of the parameter* δ *and the function* α.

Estimate (51) is fully optimal, at least for a geometrically conforming decomposition, and the possibly high ratios between the different values of the α_k are correctly taken into account by the weighted norms.
Also, the constant $\sqrt{\alpha_{\max}}$ seems unavoidable, however this is negligible since the data are most often much more regular than the solution due to the discontinuity of α.

Estimating the error on the pressure is now easy.

Theorem 15 *If the assumptions of Theorem 8 are satisfied, the following error estimate holds between the pressure p of problem* (5) *and the pressure p_δ of problem* (19) *:*

$$\begin{aligned}
&\|p - p_\delta\|_{\alpha^*} \\
&\le c\Bigg((1+\mu+\mu_\delta)^{\frac{1}{2}}\Big(\sum_{k=1}^{K} N_k^{-s_k}\Big(\alpha_k^{\frac{1}{2}}(\log N_k)\|\boldsymbol{u}_{|\Omega_k}\|_{H^{s_k}(\Omega_k)^2} + \alpha_k^{-\frac{1}{2}}\|p_{|\Omega_k}\|_{H^{s_k+1}(\Omega_k)}\Big)\Big) \\
&\qquad + \sqrt{\alpha_{\max}}\Big(\Big(\sum_{k=1}^{K} N_k^{-2\sigma_k}\|\boldsymbol{f}_{|\boldsymbol{\Omega_k}}\|^2_{H^{\sigma_k}(\Omega_k)^2}\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^{K} N_k^{-2\tau}\sum_{j=1}^{4}\|g\|^2_{H^\tau(\partial\Omega)}\Big)^{1/2}\Big)\Bigg),
\end{aligned} \tag{52}$$

where the constant c is independent of the parameter δ and the function α.

Proof. From the inf-sup condition (27), we derive that, for any q_δ in $\mathbb{M}_\delta$,

$$\beta\,\|p_\delta - q_\delta\|_{\alpha*} \le \sup_{\boldsymbol{v}_\delta \in \mathbb{X}_\delta} \frac{b_\delta(\boldsymbol{v}_\delta, p_\delta - q_\delta)}{\|\boldsymbol{v}_\delta\|_\alpha}. \tag{53}$$

we first use the discrete problem (19) :

$$b_\delta(\boldsymbol{v}_\delta, p_\delta - q_\delta) = ((\alpha\boldsymbol{f}, \boldsymbol{v}_\delta))_\delta + a_\delta(\boldsymbol{u}_\delta, \boldsymbol{v}_\delta) - b_\delta(\boldsymbol{v}_\delta, q_\delta).$$

Next, we apply equation (5) to the function $\boldsymbol{v}_\delta$, integrate by parts and add it to the previous line. This yields

$$\begin{aligned}
b_\delta(\boldsymbol{v}_\delta, p_\delta - q_\delta) = &\sum_{k=1}^{K}\alpha_k\int_{\Omega_k}(\boldsymbol{u}-\boldsymbol{u}_\delta)(\boldsymbol{x})\cdot\boldsymbol{v}_\delta(\boldsymbol{x})\,d\boldsymbol{x} \\
&+ \sum_{k=1}^{K}\alpha_k\int_{\Omega_k}\boldsymbol{u}_\delta(\boldsymbol{x})\cdot\boldsymbol{v}_\delta(\boldsymbol{x})\,d\boldsymbol{x} - ((\alpha\boldsymbol{u}_\delta, \boldsymbol{v}_\delta))_\delta \\
&+ \int_\Omega \boldsymbol{v}_\delta(\boldsymbol{x})\cdot\mathbf{grad}\,(p-q_\delta)(\boldsymbol{x})\,d\boldsymbol{x} \\
&+ ((\alpha\boldsymbol{f}, \boldsymbol{v}_\delta))_\delta - \sum_{k=1}^{K}\alpha_k\int_{\Omega_k}\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{v}_\delta(\boldsymbol{x})\,d\boldsymbol{x} \\
&+ b(\boldsymbol{v}_\delta, q_\delta) - b_\delta(\boldsymbol{v}_\delta, q_\delta)
\end{aligned} \tag{54}$$

Using the same arguments as in the estimation of terms issued from numerical integration together with a triangle inequality yields

$$\|p-q_\delta\|_{\alpha^*} \le c\Bigg(\|\boldsymbol{u}-\boldsymbol{u}_\delta\|_\alpha + \sup_{\boldsymbol{v}_\delta\in\mathbb{X}_\delta} \frac{(a_\alpha - a_\alpha^\delta)(\boldsymbol{u}_\delta,\boldsymbol{v}_\delta)}{\|\boldsymbol{v}_\delta\|_\alpha} \\ + \|p-q_\delta\|_{\alpha*} + \sup_{\boldsymbol{v}_\delta\in\mathbb{X}_\delta} \frac{((\alpha\boldsymbol{f},\boldsymbol{v}_\delta))_\delta - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{v}_\delta(\boldsymbol{x})\,d\boldsymbol{x}}{\|\boldsymbol{v}_\delta\|_\alpha} \\ + \sup_{\boldsymbol{v}_\delta\in\mathbb{X}_\delta} \frac{b(\boldsymbol{v}_\delta,q_\delta)-b_\delta(\boldsymbol{v}_\delta,q_\delta)}{\|\boldsymbol{v}_\delta\|_\alpha}\Bigg). \quad (55)$$

All the terms in the right-hand side have been estimated previously. ■

A more explicit estimate can be deduced from the previously quoted regularity results. We refer to [6] for proof.

Corollary 16 *Assume the datum $\boldsymbol{f}$ such that each $\boldsymbol{f}_{|\Omega_k}$, $1lek \le K$, belongs to $H^{\sigma_k}(\Omega_k)^2$, $\sigma_k > 1$, and the datum g belong to $H^\tau(\partial\Omega)$, $\tau > \frac{1}{2}$. Then, the following error estimate holds between the solution $(\boldsymbol{u},p)$ of problem (5) and the solution $(\boldsymbol{u}_\delta,p_\delta)$ of problem (19) :*

$$\|\boldsymbol{u}-\boldsymbol{u}_\delta\|_\alpha + \|p-p_\delta\|_{\alpha^*} \le cE_k \sum_{k=1}^K (\|\boldsymbol{f}_{|\Omega_k}\|_{H^{\sigma_k}(\Omega_k)^2} + \sum_{j=1}^4 \|g\|_{H^\tau(\partial\Omega)}), \quad (56)$$

with

$$E_k = \begin{cases} \sup\{N_k^{-4}(\log N_k)^{\frac{3}{2}}, N_k^{-\sigma_k}\}, & \text{if } \overline{\Omega}_k \text{ contains a corner but no nonconvex corner of } \Omega, \\ \sup\{N_k^{-\frac{4}{3}}(\log N_k)^{\frac{1}{2}}, N_k^{-\sigma_k}\}, & \text{if } \overline{\Omega}_k \text{ contains a nonconvex corner of } \Omega, \\ N_k^{-\sigma_k}, & \text{if } \overline{\Omega}_k \text{ contains no corner of } \Omega. \end{cases}$$

5 Extension to piecewise smooth coefficients

We are now interested in the case where the α_k are no longer constants but are smooth functions. From now on, we do not take into account the local ratios $\alpha^k_{\max}/\alpha^k_{\min}$, where $\alpha^k_{\min}$ and $\alpha^k_{\max}$, $1 \le k \le K$, are introduced in (9), but only the global one $\alpha_{\max}/\alpha_{\min}$.
The discrete problem relies on the same space $\mathbb{X}_\delta$ and $\mathbb{M}_\delta$, and on the same discrete product $((.,.))_\delta$.
If the function $\boldsymbol{f}$ has continuous restrictions to all $\overline{\Omega}_k$, $1 \le k \le K$, and the datum g fas continuous restrictions to $\overline{\partial\Omega}$, it reads

Find $(\boldsymbol{u}_\delta, p_\delta) \in \mathbb{X}_\delta \times \mathbb{M}_\delta$ such that

$$\begin{cases} \forall \boldsymbol{v} \in \mathbb{X}_\delta,\ a^\delta_\alpha(\boldsymbol{u}_\delta, \boldsymbol{v}_\delta) + b_\delta(\boldsymbol{v}_\delta, p_\delta) & = \quad ((\alpha \boldsymbol{f}, \boldsymbol{v}_\delta))_\delta, \\ \forall q_\delta \in \mathbb{M}_\delta,\ b_\delta(\boldsymbol{u}_\delta, q_\delta) & = \quad ((g, q_\delta))^{\partial\Omega}_\delta, \end{cases} \tag{57}$$

where the bilinear form $a_\delta(.,.)$ is now defined by

$$a^\delta_\alpha(\boldsymbol{u}_\delta, \boldsymbol{v}_\delta) = \sum_{k=1}^K ((\alpha_k \boldsymbol{u}_\delta, \boldsymbol{v}_\delta))^k_\delta, \tag{58}$$

we conserve the bilinear form $b_\delta(.,.)$, and we define

$$((\alpha \boldsymbol{f}, \mathbf{v}_\delta))_\delta = \sum_{k=1}^K ((\alpha_k \boldsymbol{f}, \boldsymbol{v}_\delta))^k_\delta.$$

We decide here to define the "broken"energy norm by

$$\|q\|_{\alpha^*} = \left(\sum_{k=1}^K (\alpha^k_{\max})^{-1} |q|^2_{H^1(\Omega_k)} \right)^{\frac{1}{2}}. \tag{59}$$

The statements of Lemmas 5 and 4 are still valid in this case (with the constants 9 in (21) and 1 in (22) replaced by appropriate constants only depending on the ratios $\alpha^k_{\max}/\alpha^k_{\min}$. This yields the well-posedness of problem (57).

Proposition 17 *For any datum $\boldsymbol{f}$ such that each $\boldsymbol{f}_{|\Omega_k}$, $1 \le k \le K$, is continuous on $\overline{\Omega}_k$, and the datum g is continuous on $\overline{\partial\Omega}$, problem* (57) *has a unique solution $(\boldsymbol{u}_\delta, p_\delta)$ in $\mathbb{X}_\delta \times \mathbb{M}_\delta$. Moreover, there exists a constant c independent of δ such that this solution satisfies*

$$\|\boldsymbol{u}_\delta\|_\alpha + \|p_\delta\|_{\alpha*} \le c\sqrt{\alpha_{\max}} (\|\mathcal{I}_\delta \boldsymbol{f}\|_\alpha + \|\mathcal{I}^{\partial\Omega}_\delta g\|_{L^2(\partial\Omega)}). \tag{60}$$

Proving the error estimates is slightly more complex. Only the term

$$\sup_{\boldsymbol{w}_\delta \in \mathbb{X}_\delta} \frac{(a_\alpha - a^\delta_\alpha)(\boldsymbol{v}_\delta, \boldsymbol{w}_\delta)}{\|\mathbf{w}_\delta\|_\alpha},$$

and

$$\sup_{\boldsymbol{w}_\delta \in \mathbb{X}_\delta} \frac{((\alpha \boldsymbol{f}, \boldsymbol{w}_\delta))_\alpha - \sum_{k=1}^K \int_{\Omega_k} \alpha_k \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{w}_\delta(\boldsymbol{x})\, d\boldsymbol{x}}{\|\boldsymbol{w}_\delta\|_\alpha},$$

requires some further attention.
Let $\mathbb{Z}_{\delta'}$ be the analogue of the space $\mathbb{Z}_\delta$ introduced in (**??**), for the K-tuple δ' made of the N'_k, where each N'_k is equal to the integral part of $(N_k-1)/2$.

Lemma 18 *If the functions α_k, $1 \le k \le K$, belong to $H^{\varsigma_k}(\Omega_k)$, $\varsigma_k > 3/2$, the following estimate holds for any $\boldsymbol{v}_\delta$ in $\mathbb{X}_\delta$*

$$\begin{aligned}
&\sup_{\boldsymbol{w}_\delta \in \mathbb{X}_\delta} \frac{(a_\alpha - a_\alpha^\delta)(\boldsymbol{v}_\delta, \boldsymbol{w}_\delta)}{\|\boldsymbol{w}_\delta\|_\alpha} \\
&\qquad \le c\Bigg(\Big(\sum_{k=1}^K (\alpha_{\max}^{k-2})^{-1} N_k^{2(1-\varsigma_k)} (\log N_k) \|\alpha_k\|^2_{H^{\varsigma_k}(\Omega_k)}\Big)^{1/2} \Big(\|\boldsymbol{u}\|_\alpha + \|\boldsymbol{u}-\boldsymbol{v}_\delta\|_\alpha\Big) \\
&\qquad\qquad + \|\boldsymbol{u}-\boldsymbol{v}_\delta\|_\alpha + \inf_{\boldsymbol{z}_{\delta'} \in \mathbb{Z}_{\delta'}} \|\boldsymbol{u}-\boldsymbol{z}_{\delta'}\|_\alpha\Bigg).
\end{aligned}$$

Proof. For any functions $\boldsymbol{v}_\delta$ and $\boldsymbol{w}_\delta$ in $\mathbb{X}_\delta$, we have

$$(a_\alpha - a_\alpha^\delta)(\boldsymbol{v}_\delta, \boldsymbol{w}_\delta) = \sum_{k=1}^K \left(\int_{\Omega_k} \alpha_k(\boldsymbol{x}) \boldsymbol{v}_\delta(\boldsymbol{x}) \cdot \boldsymbol{w}_\delta(\boldsymbol{x})\, d\boldsymbol{x} - (((\mathcal{I}_\delta \alpha_k)\boldsymbol{v}_\delta, \boldsymbol{w}_\delta))^k_{N_k} \right).$$

On each Ω_k, we introduce the image $\boldsymbol{z}_{k\delta'}$ of $\boldsymbol{u}$ by the orthogonal projection operator from $H^1(\Omega_k)$ onto $\mathbb{P}_{N'_k}(\Omega_k)$, together with an approximation $\alpha_{k\delta'}$ of α_k in $\mathbb{P}_{N'_k}(\Omega_k)$. By adding and subtracting the term

$$\int_{\Omega_k} \alpha_{k\delta'}(\boldsymbol{x}) \boldsymbol{z}_{k\delta'}(\boldsymbol{x}) \cdot \boldsymbol{w}_\delta(\boldsymbol{x})\, d\boldsymbol{x} = (((\mathcal{I}_\delta \alpha_{k\delta'})\boldsymbol{v}_{k\delta'}, \boldsymbol{w}_\delta))^k_{N_k},$$

we have to bound the quantities

$$(\alpha_{\max}^{k-1/2})^{-1} \|\alpha_k \boldsymbol{v}_\delta - \alpha_{k\delta'} \boldsymbol{z}_{k\delta'}\|_{L^2(\Omega_k)^d} \text{ and } (\alpha_{\max}^{k-1/2})^{-1} \|\mathcal{I}_\delta((\mathcal{I}_\delta \alpha_k)\boldsymbol{v}_\delta - \alpha_{k\delta'} \boldsymbol{z}_{k\delta'})\|_{L^2(\Omega_k)^d}.$$

The first is bounded by

$$\begin{aligned}
c(\alpha_{\max}^{k-1/2})^{-1} &(\|\boldsymbol{u}-\boldsymbol{v}_\delta\|_{L^2(\Omega_k)^d} + \|\boldsymbol{u}-\boldsymbol{v}_{k\delta'}\|_{L^2(\Omega_k)^d}) \\
&\qquad + c'(\alpha_{\max}^{k-1/2})^{-1} \|\alpha_k - \alpha_{k\delta'}\|_{L^\infty(\Omega_k)} \|\boldsymbol{u}\|_{L^2(\Omega_k)^d}.
\end{aligned} \tag{61}$$

Moreover let us recall from [7, Eq. (13.28)] that

$$\forall v_M \in \mathbb{P}_M(\Omega_k),\ \|\mathcal{I}_\delta v_M\|_{L^2(\Omega_k)} \le c \left(1 + \frac{M}{N_k}\right)^2 \|v_M\|_{L^2(\Omega_k)}.$$

So, since the restriction of $(\mathcal{I}_\delta \alpha_k)\boldsymbol{v}_\delta - \alpha_{k\delta'}\boldsymbol{z}_{k\delta'}$ to each Ω_k belongs to $\mathbb{P}_{2N_k}(\Omega_k)$, the second term is bounded by a constant times the quantities in (61) plus

$$c''(\alpha_{\max}^{k-1/2})^{-1}\|\alpha_k - \mathcal{I}_\delta\alpha_k\|_{L^\infty(\Omega_k)}(\|\boldsymbol{u}\|_{L^2(\Omega_k)^2} + \|\boldsymbol{u}-\boldsymbol{v}_\delta\|_{L^2(\Omega_k)^2}).$$

So, when taking $\alpha_{k\delta'} = (\mathcal{I}_{\delta'}\alpha)_{|\Omega_k}$ (with obvious notation), the desired estimate follows from

$$|\alpha_k - \mathcal{I}_\delta\alpha_k\|_{L^\infty(\Omega_k)} \leq cN_k^{1-\tau_k}(\log N_k)^{1/2}\|\alpha_k\|_{H^{\tau_k}(\Omega_k)},$$

which can be derived from [7, Sec. 14] combined with Gagliardo-Nirenberg inequality. ■

We can now conclude with the error estimates, which are the same as in Section 3 with a further term involving the regularity of the α_k.

Theorem 19 *Assume that the functions α_k, $1 \leq k \leq K$, belong to $H^{\tau_k}(\Omega_k)$, $\tau_k > 3/2$. If the solution $(\boldsymbol{u}, p)$ of problem (5) is such that its restriction to each $(\boldsymbol{u}_{|\Omega_k}, p_{|\Omega_k})$, $1 \leq k \leq K$ belongs to $H^{s_k}(\Omega_k)^2 \times H^{s_k+1}(\Omega_k)$, $s_k \geq 0$, and if the function $\boldsymbol{f}$ is such that its restriction to each $\boldsymbol{f}_{|\Omega_k}$, belongs to $H^{\sigma_k}(\Omega_k)$, for integer $\sigma_k > 1$, the following error estimate holds between this solution and the solution $(\boldsymbol{u}_\delta, p_\delta)$ of problem (57)*

$$\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_\delta\|_\alpha + \|p-p_\delta\|_{\alpha^*} \\
&\quad\leq c\Bigg((1+\mu+\mu_\delta)\sum_{k=1}^K N_k^{-2s_k}\Big(\alpha_k(\log N_k)\|\boldsymbol{u}_{|\Omega_k}\|^2_{H^{s_k}(\Omega_k)} + N_k^{-2s_k}\alpha_k^{-1}\|p_{|\Omega_k}\|^2_{H^{s_k+1}(\Omega_k)}\Big)^{1/2} \\
&\qquad\qquad + \Big(\sum_{k=1}^K \alpha_{\min}^{k-2}N_k^{2(1-\varsigma_k)}(\log N_k)\|\alpha_k\|^2_{H^{\varsigma_k}(\Omega_k)}\Big)^{1/2}\|\boldsymbol{u}\|_\alpha \\
&\qquad\qquad + \sqrt{\alpha_{\max}}\Big(\sum_{k=1}^K N_k^{-2\sigma_k}\|\boldsymbol{f}_{|\Omega_k}\|^2_{H^{\sigma_k}(\Omega_k)^2} + \sum_{k=1}^K N_k^{-2\tau}\sum_{j=1}^4\|g\|^2_{H^\tau(\partial\Omega)}\Big)^{1/2}\Bigg),
\end{aligned}\tag{62}$$

where the constant c is indépendent of parameter δ and the function α.

There the following improved estimate also holds for a conforming decomposition.

Corollary 20 *If the decomposition (10) is conforming and if the assumptions of Theorem 19 are satisfied, the following error estimate holds between the solution $(\boldsymbol{u}, p)$ of problem (5) and the solution $(\boldsymbol{u}_\delta, p_\delta)$ of problem (57)*

$$\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_\delta\|_\alpha + \|p-p_\delta\|_{\alpha^*} \\
&\quad\leq c\Bigg((1+\mu)\Big(\sum_{k=1}^K N_k^{-2s_k}\Big(\alpha_k\|\boldsymbol{u}_{|\Omega_k}\|^2_{H^{s_k}(\Omega_k)^2} + \alpha_k^{-1}\|p_{|\Omega_k}\|^2_{H^{s_k+1}(\Omega_k)}\Big)^{1/2} \\
&\qquad\qquad + \Big(\sum_{k=1}^K(\alpha_{\min}^{k-2}N_k^{2(1-\varsigma_k)}(\log N_k)\|\alpha_k\|^2_{H^{\varsigma_k}(\Omega_k)}\Big)^{1/2}\|\boldsymbol{u}\|_\alpha \\
&\qquad\qquad + \sqrt{\alpha_{\max}}\Big(\sum_{k=1}^K N_k^{-2\sigma_k}\|\mathbf{f}_{|\Omega_k}\|^2_{H^{\sigma_k}(\Omega_k)^2} + \sum_{k=1}^K N_k^{-2\tau}\sum_{j=1}^4\|g\|^2_{H^\tau(\partial\Omega)}\Big)^{1/2}\Bigg),
\end{aligned}\tag{63}$$

where the constant c is independent of parameter δ and the function α.

Since the functions α_k are assumed to be smooth, the convergence order is exactly the same as in Section 3.

6 Numerical experiments

First, we briefly describe the implementation of the discrete problem. The unknowns are the values of the solution $(\boldsymbol{u}_\delta, p_\delta)$ at the nodes $(\xi_{ki}^{(x)}, \xi_{kj}^{(y)})$, $0 \le i, j \le N_k$, $1 \le k \le K$, which either are inside the Ω_k or are corners of the Ω_k that do not belong to $\partial\Omega$ or are inside the mortars γ_m. Let (U, P) denote the vector made of these values. Then conditions (13) can be expressed in the following way : there exists a rectangular matrix $\mathbf{Q}$ such that the vector $\widetilde{P} = \mathbf{Q}P$ is made of K blocks P_k, $1 \le k \le K$, and each P_k is made of the values of p_δ at all nodes $(\xi_{ki}^{(x)}, \xi_{kj}^{(y)})$, $0 \le i, j \le N_k$, $1 \le k \le K$.
The problem (19) is now equivalent to the following square linear

$$\begin{cases} AU + B\mathbf{Q}\tilde{P} & = & F, \\ \\ \mathbf{Q}^T B^T U & = & \mathbf{Q}^T G, \end{cases} \tag{64}$$

where $\mathbf{Q}^T$ stands for the transposed matrix of $\mathbf{Q}$. The matrix A is fully diagonal, its diagonal terms are the $\rho_{ik}^{(x)} \rho_{jk}^{(y)}$ according to the dimension. The matrix B is only block-diagonal, with K blocks B_k on the diagonal, one for each Ω_k. Since, system (64) is solved via the conjugate gradient algorithm.
Our first experiments concern the simple geometry where Ω is a rectangle divided into two squares

$$\Omega =]-1, 1[\times]0, 1[, \qquad \Omega_1 =]-1, 0[\times]0, 1[, \qquad \Omega_2 =]0, 1[\times]0, 1[,$$

when the corresponding pair (α_1, α_2) of values of α runs through $(1, 10)$ and $(10^2, 10^3)$.
In Fig. 1, the error are presented for the discretization without domain decomposition.

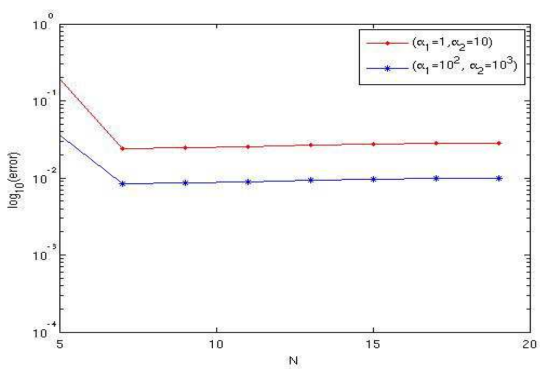

FIGURE 1 – Error curves

We now consider the case of non-conforming decomposition see Figure 2. The domain is

$$\Omega =]-1, 1[^2,$$

partitioned into three subdomains

$$\Omega_1 =]-1, 0[\times]0, 1[, \quad \Omega_2 =]0, 1[\times]0, 1[, \quad \Omega_3 =]-1, 1[\times]-1, 0[.$$

The mortars are chosen as

$$\gamma_1 = \{0\}\times]0, 1[, \quad \gamma_2 =]-1, 1[\times\{0\}.$$

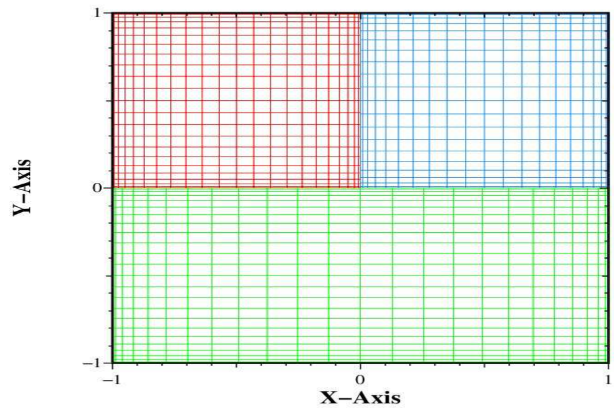

FIGURE 2 – The nonmatching grids for a nonconforming decomposition with $N_1 = 24, N_2 = 22, N_3 = 20$.

The coefficients α_k are equal to

$$\alpha_1 = 1,\ \ \alpha_2 = 10,\ \ \alpha_3 = 100.$$

We use our spectral method to copppute an approximation of the analytical solution (u, p) given by

$$\boldsymbol{u}(x, y) = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ \\ -\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p(x, y) = \sin(\pi x)\cos(\pi y). \tag{65}$$

In Figure 3 are plotted, the curves of the errors $\|\boldsymbol{u} - \boldsymbol{u}_\delta\|_\alpha$ and $\|p - p_\delta\|_{\alpha*}$ for both cases as a function of N. For the smooth solution, a linear or logarithmic scale is used and we observe that the exponential decaying of the error is preserved despite the nonconforming domain decompsotion. For the nonsmooth soution rather a full logarithmic scale is adopted, we observe the good convergence of the discretization. This solution is given by

$$\boldsymbol{u}(x, y) = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ \\ -\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p(x, y) = ((x-1)^2 + (y-1)^2)^{\frac{5}{4}}.$$

The slopes of the curves are -2.1 and -4.5, so they are better than the theoretical prediction (we refer to [1] for the first observation of this superconvergence phenomenon).
We represente in Figure 4 and 5 the solution with N equal to 80.

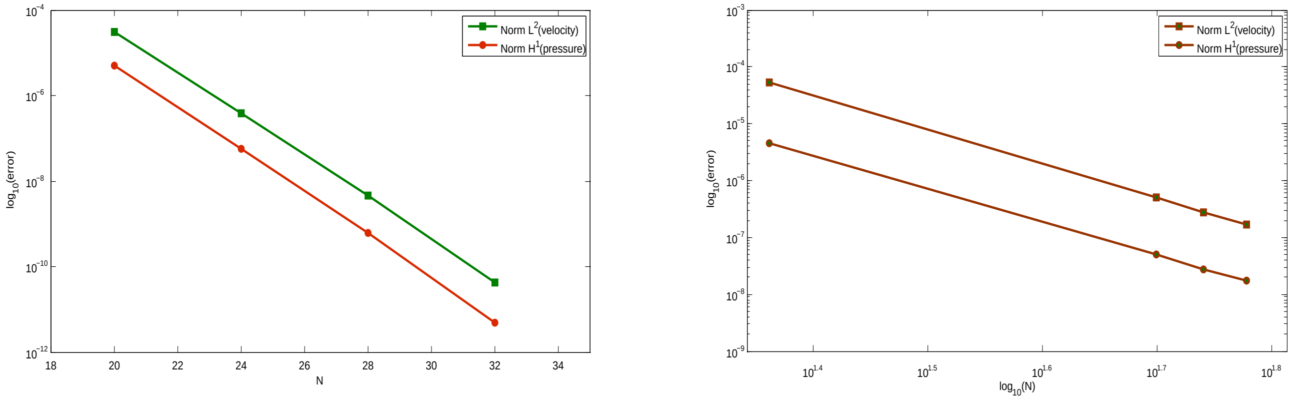

FIGURE 3 – The error curves for an analytical solution (left panel) and a nonsmooth solution (right panel).

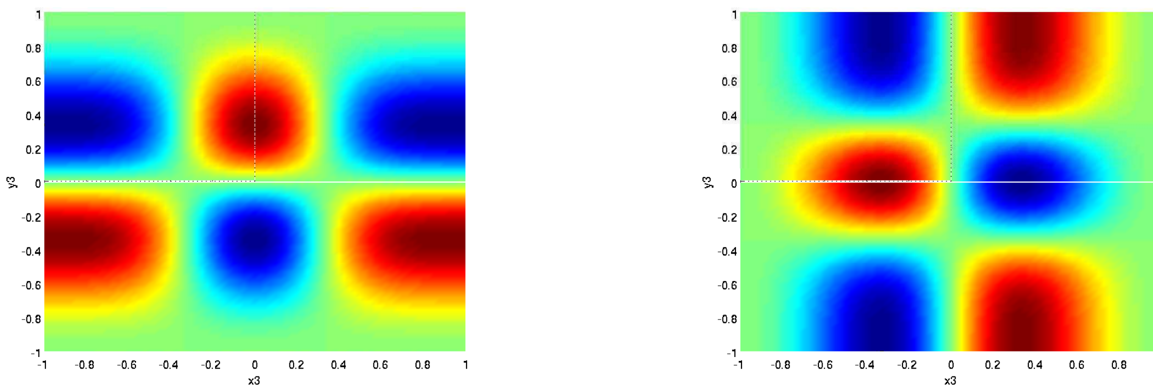

FIGURE 4 – The isovalues of the two components of the velocity.

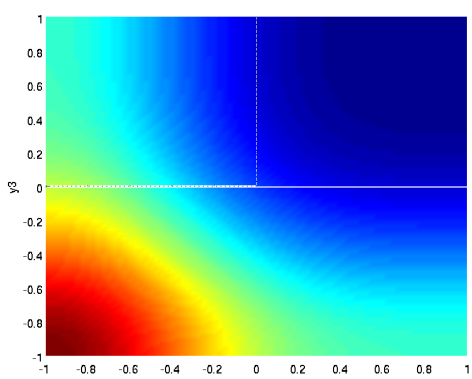

FIGURE 5 – The isovalues of the pressure.

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